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► To cite this version:

Bachir Bekka, Alexander Lubotzky. Lattices with and lattices without spectral gap. Groups, Geometry, and Dynamics, 2011, 5 (2), pp.251-264. 10.4171/GGD/126 . hal-00564631

HAL Id: hal-00564631

<https://hal.science/hal-00564631>

Submitted on 9 Feb 2011

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Lattices with and lattices without spectral gap

Bachir Bekka and Alexander Lubotzky *

September 10, 2009

For Fritz Grunewald on his 60th birthday

Abstract

Let $G = \mathbf{G}(\mathbf{k})$ be the \mathbf{k} -rational points of a simple algebraic group \mathbf{G} over a local field \mathbf{k} and let Γ be a lattice in G . We show that the regular representation $\rho_{\Gamma \backslash G}$ of G on $L^2(\Gamma \backslash G)$ has a spectral gap, that is, the restriction of $\rho_{\Gamma \backslash G}$ to the orthogonal of the constants in $L^2(\Gamma \backslash G)$ has no almost invariant vectors. On the other hand, we give examples of locally compact simple groups G and lattices Γ for which $L^2(\Gamma \backslash G)$ has no spectral gap. This answers in the negative a question asked by Margulis [Marg91, Chapter III, 1.12]. In fact, G can be taken to be the group of orientation preserving automorphisms of a k -regular tree for $k > 2$.

1 Introduction

Let G be a locally compact group. Recall that a unitary representation π of G on a Hilbert space \mathcal{H} has almost invariant vectors if, for every compact subset Q of G and every $\varepsilon > 0$, there exists a unit vector $\xi \in \mathcal{H}$ such that $\sup_{x \in Q} \|\pi(x)\xi - \xi\| < \varepsilon$. If this holds, we also say that the trivial representation 1_G is weakly contained in π .

Recall that a lattice Γ in G is a discrete subgroup such that there exists a finite G -invariant regular Borel measure μ on $\Gamma \backslash G$. Denote by $\rho_{\Gamma \backslash G}$ the unitary representation of G given by right translation on the Hilbert space

*This research was supported by a grant from the ERC

$L^2(\Gamma \backslash G, \mu)$ of the square integrable measurable functions on $\Gamma \backslash G$. The subspace $\mathbb{C}1_{\Gamma \backslash G}$ of the constant functions on $\Gamma \backslash G$ is G -invariant as well as its orthogonal complement

$$L_0^2(\Gamma \backslash G) = \left\{ \xi \in L^2(\Gamma \backslash G) : \int_{\Gamma \backslash G} \xi(x) d\mu(x) = 0 \right\}.$$

Denote by $\rho_{\Gamma \backslash G}^0$ the restriction of $\rho_{\Gamma \backslash G}$ to $L_0^2(\Gamma \backslash G, \mu)$. We say that $\rho_{\Gamma \backslash G}$ (or $L^2(\Gamma \backslash G, \mu)$) has a *spectral gap* if $\rho_{\Gamma \backslash G}^0$ has no almost invariant vectors. (In [Marg91, Chapter III., 1.8], Γ is then called weakly cocompact.) It is well-known that $L^2(\Gamma \backslash G)$ has a spectral gap when Γ is cocompact in G (see [Marg91, Chapter III, 1.10]). Margulis (*op.cit.*, 1.12) asks whether this result holds more generally when Γ is a subgroup of finite covolume.

The goal of this note is to prove the following results:

Theorem 1 *Let \mathbf{G} be a simple algebraic group over a local field \mathbf{k} and $G = \mathbf{G}(\mathbf{k})$, the group of \mathbf{k} -rational points in \mathbf{G} . Let Γ be a lattice in G . Then the unitary representation $\rho_{\Gamma \backslash G}$ on $L^2(\Gamma \backslash G)$ has a spectral gap.*

Theorem 2 *For an integer $k > 2$, let X be the k -regular tree and $G = \text{Aut}(X)$. Then G contains a lattice Γ for which the unitary representation $\rho_{\Gamma \backslash G}$ on $L^2(\Gamma \backslash G)$ has no spectral gap.*

So, Theorem 2 answers in the negative Margulis' question mentioned above.

Theorem 1 is known in case $\mathbf{k} = \mathbf{R}$ ([Bekk98]). It holds, more generally, when G is a real Lie group ([BeCo08]). Observe also that when $\mathbf{k} - \text{rank}(\mathbf{G}) \geq 2$, the group G has Kazhdan's Property (T) (see [BHV]) and Theorem 1 is clear in this case. When \mathbf{k} is non-archimedean with characteristic 0, every lattice Γ in $\mathbf{G}(\mathbf{k})$ is uniform (see [Serr, p.84]) and hence the result holds as mentioned above. By way of contrast, G has many non uniform lattices when the characteristic of \mathbf{k} is non zero (see [Serr] and [Lubo91]). So, in order to prove Theorem 1, it suffices to consider the case where the characteristic of \mathbf{k} is non-zero and where $\mathbf{k} - \text{rank}(\mathbf{G}) = 1$.

Recall that when \mathbf{k} is non-archimedean and $\mathbf{k} - \text{rank}(\mathbf{G}) = 1$, the group $\mathbf{G}(\mathbf{k})$ acts by automorphisms on the associated Bruhat-Tits tree X (see [Serr]). This tree is either the k -regular tree X_k (in which every vertex has constant degree k) or is the bi-partite bi-regular tree X_{k_0, k_1} (where every vertex has either degree k_0 or degree k_1 and where all neighbours of a vertex

of degree k_i have degree k_{1-i}). The proof of Theorem 1 will use the special structure of a fundamental domain for the action of Γ on X as described in [Lubo91] (see also [Ragh89] and [Baum03]).

Theorems 1 and 2 provide a further illustration of the different behaviour of general tree lattices as compared to lattices in rank one simple Lie groups over local fields; for more on this topic, see [Lubo95].

The proofs of Theorems 1 and 2 will be given in Sections 3 and 4; they rely in a crucial way on Proposition 6 from Section 2, which relates the existence of a spectral gap with expander diagrams. In turn, Proposition 6 is based, much in the spirit of [Broo81], on analogues for diagrams proved in [Mokh03] and [Morg94] of the inequalities of Cheeger and Buser between the isoperimetric constant and the bottom of the spectrum of the Laplace operator on a Riemannian manifold (see Proposition 5). This connection between the combinatorial expanding property and representation theory is by now a very popular theme; see [Lubo94] and the references therein. While most applications in this monograph are from representation theory to combinatorics, we use in the current paper this connection in the opposite direction: the existence or absence of a spectral gap is deduced from the existence of an expanding diagram or of a non-expanding diagram, respectively.

2 Spectral gap and expander diagrams

We first show how the existence of a spectral gap for groups acting on trees is related with the bottom of the spectrum of the Laplacian for an associated diagram.

A graph X consists of a set of vertices VX , a set of oriented edges EX , a fix-point free involution $- : EX \rightarrow EX$, and end point mappings $\partial_i : EX \rightarrow VX$ for $i = 0, 1$ such that $\partial_i(\bar{e}) = \partial_{1-i}(e)$ for all $e \in EX$. Assume that X is locally finite, that is, for every $x \in VX$, the degree $\deg(x)$ of x is finite, where $\deg(x)$ is the cardinality of the set

$$\partial_0^{-1}(x) = \{e \in EX : \partial_0(e) = x\}.$$

The group $\text{Aut}(X)$ of automorphisms of the graph X is a locally compact group in the topology of pointwise convergence on X , for which the stabilizers of vertices are compact open subgroups.

We will consider infinite graphs called diagrams of finite volume. An *edge-indexed graph* (D, i) is a graph D equipped with a function $i : ED \rightarrow \mathbf{R}^+$

(see [BaLu01, Chapter 2]). A measure μ for an edge-indexed graph (D, i) is a function $\mu : VD \cup ED \rightarrow \mathbf{R}^+$ with the following properties (see [Mokh03] and [BaLu01, 2.6]):

- $i(e)\mu(\partial_0 e) = \mu(e)$
- $\mu(e) = \mu(\bar{e})$ for all $e \in VD$, and
- $\sum_{x \in VD} \mu(x) < \infty$.

Following [Morg94], we will say that $D = (D, i, \mu)$ is a *diagram of finite volume*. The in-degree $\text{indeg}(x)$ of a vertex $x \in VD$ is defined by

$$\text{indeg}(x) = \sum_{e \in \partial_0^{-1}(x)} i(e) = \sum_{e \in \partial_0^{-1}(x)} \frac{\mu(e)}{\mu(x)}.$$

The diagram D is k -regular if $\text{indeg}(x) = k$ for all $x \in VD$.

Let $D = (D, i, \mu)$ be a connected diagram of finite volume. Observe that μ is determined, up to a multiplicative constant, by the weight function i . Indeed, fix $x_0 \in VD$ and set $\Delta(e) = i(e)/i(\bar{e})$ for $e \in ED$. Then

$$\mu(\partial_1 e) = \frac{\mu(\bar{e})}{i(\bar{e})} = \frac{\mu(e)}{i(\bar{e})} = \mu(\partial_0 e)\Delta(e)$$

for every $e \in ED$. Hence $\mu(x) = \Delta(e_1)\Delta(e_2)\dots\Delta(e_n)\mu(x_0)$ for every path (e_1, e_2, \dots, e_n) from x_0 to $x \in VD$.

Let $D = (D, i, \mu)$ be a diagram of finite volume. An inner product is defined for functions on VD by

$$\langle f, g \rangle = \sum_{x \in VD} f(x)\overline{g(x)}\mu(x).$$

The Laplace operator Δ on functions f on VD is defined by

$$\Delta f(x) = f(x) - \frac{1}{\text{indeg}(x)} \sum_{e \in \partial_0^{-1}(x)} \frac{\mu(e)}{\mu(x)} f(\partial_1(e)).$$

The operator Δ is a self-adjoint positive operator on $L^2(VD)$. Let

$$L_0^2(VD) = \{f \in L^2(VD) : \langle f, 1_{VD} \rangle = 0\}$$

and set

$$\lambda(D) = \inf_f \langle \Delta f, f \rangle,$$

where f runs over the unit sphere in $L_0^2(VD)$. Observe that

$$\lambda(D) = \inf \{ \lambda : \lambda \in \sigma(\Delta) \setminus \{0\} \},$$

where $\sigma(\Delta)$ is the spectrum of Δ .

Let now X be a locally finite tree, and let G be a closed subgroup of $\text{Aut}(X)$. Assume that G acts with finitely many orbits on X . Let Γ be a discrete subgroup of G acting without inversion on X . Then the quotient graph $\Gamma \backslash X$ is well-defined. Since Γ is discrete, for every vertex x and every edge e , the stabilizers Γ_x and Γ_e are finite. Moreover, Γ is a lattice in G if and only if Γ is a lattice in $\text{Aut}(X)$ and this happens if and only if

$$\sum_{x \in D} \frac{1}{|\Gamma_x|} < \infty,$$

where D is a fundamental domain of Γ in X (see [Serr]). The quotient graph $\Gamma \backslash X \cong D$ is endowed with the structure of an edge-indexed graph given by the weight function $i : ED \rightarrow \mathbf{R}^+$ where $i(e)$ is the index of Γ_e in Γ_x for $x = \partial_0(e)$. A measure $\mu : VD \cup ED \rightarrow \mathbf{R}^+$ is defined by

$$\mu(x) = \frac{1}{|\Gamma_x|} \quad \text{and} \quad \mu(e) = \frac{1}{|\Gamma_e|}$$

for $x \in VD$ and $e \in ED$. Observe that $\mu(VD) = \sum_{x \in D} 1/|\Gamma_x| < \infty$. So, $D = (D, i, \mu)$ is a diagram of finite volume.

Let G be a group acting on a tree X . As in [BuMo00, 0.2], we say that the action of G on X is *locally ∞ -transitive* if, for every $x \in VX$ and every $n \geq 1$, the stabilizer G_x of x acts transitively on the sphere $\{y \in X : d(x, y) = n\}$.

Proposition 3 *Let X be either the k -regular tree X_k or the bi-partite bi-regular tree X_{k_0, k_1} for $k \geq 3$ or $k_0 \geq 3$ and $k_1 \geq 3$. Let G be a closed subgroup of $\text{Aut}(X)$. Assume that the following conditions are both satisfied:*

- *G acts transitively on VX in the case $X = X_k$ and G acts transitively on the set of vertices of degree k_0 as well as on the set of vertices of degree k_1 in the case $X = X_{k_0, k_1}$;*

- the action of G on X is locally ∞ -transitive.

Let Γ be a lattice in G and let $D = \Gamma \backslash X$ be the corresponding diagram of finite volume. The following properties are equivalent:

- (i) the unitary representation $\rho_{\Gamma \backslash G}$ on $L^2(\Gamma \backslash G)$ has a spectral gap;
- (ii) $\lambda(D) > 0$.

For the proof of this proposition, we will need a few general facts. Let G be a second countable locally compact group and U a compact subgroup of G . Let $C_c(U \backslash G/U)$ be the space of continuous functions $f : G \rightarrow \mathbf{C}$ which have compact support and which are constant on the double cosets UgU for $g \in G$.

Fix a left Haar measure μ on G . Recall that $L^1(G, \mu)$ is a Banach algebra under the convolution product, the L^1 -norm and the involution $f^*(g) = \overline{f(g^{-1})}$; observe that $C_c(U \backslash G/U)$ is a $*$ -subalgebra of $L^1(G, \mu)$. Let π be a (strongly continuous) unitary representation of G on a Hilbert space \mathcal{H} . A continuous $*$ -representation of $L^1(G)$, still denoted by π , is defined on \mathcal{H} by

$$\pi(f)\xi = \int_G f(x)\pi(x)\xi d\mu(x), \quad f \in L^1(G), \quad \xi \in \mathcal{H}.$$

Assume that the closed subspace \mathcal{H}^U of U -invariant vectors in \mathcal{H} is non-zero. Then $\pi(f)\mathcal{H}^U \subset \mathcal{H}^U$ for all $f \in C_c(U \backslash G/U)$. In this way, a continuous $*$ -representation π_U of $C_c(U \backslash G/U)$ is defined on \mathcal{H}^U .

Proposition 4 *With the previous notation, let $f \in C_c(U \backslash G/U)$ be a function with the following properties: $f(x) \geq 0$ for all $x \in G$, $\int_G f d\mu = 1$, and the subgroup generated by the support of f is dense in G . The following conditions are equivalent:*

- (i) the trivial representation 1_G is weakly contained in π ;
- (ii) 1 belongs to the spectrum of the operator $\pi_U(f)$.

Proof Assume that 1_G is weakly contained in π . There exists a sequence of unit vectors $\xi_n \in \mathcal{H}$ such that

$$\lim_n \|\pi(x)\xi_n - \xi_n\| = 0,$$

uniformly over compact subsets of G . Let

$$\eta_n = \int_U \pi(u) \xi_n du,$$

where du denotes the normalized Haar measure on U . It is easily checked that $\eta_n \in \mathcal{H}^U$ and that

$$\lim_n \|\pi(f)\eta_n - \eta_n\| = 0.$$

Since

$$\|\eta_n - \xi_n\| \leq \int_U \|\pi(u)\xi_n - \xi_n\| du,$$

we have $\|\eta_n\| \geq 1/2$ for sufficiently large n . This shows that 1 belongs to the spectrum of the operator $\pi_U(f)$.

For the converse, assume that 1 belongs to the spectrum of $\pi_U(f)$. Hence, 1 belongs to the spectrum of $\pi(f)$, since $\pi_U(f)$ is the restriction of $\pi(f)$ to the invariant subspace \mathcal{H}^U . As the subgroup generated by the support of f is dense in G , this implies that 1_G is weakly contained in π (see [BHV, Proposition G.4.2]).

Proof of Proposition 3 We give the proof only in the case where X is the bi-regular tree X_{k_0, k_1} . The case where X is the regular tree X_k is similar and even simpler.

Let X_0 and X_1 be the subsets of X consisting of the vertices of degree k_0 and k_1 , respectively. Fix two points $x_0 \in X_0$ and $x_1 \in X_1$ with $d(x_0, x_1) = 1$. So, X_0 is the set of vertices x for which $d(x_0, x)$ is even and X_1 is the set of vertices x for which $d(x_0, x)$ is odd. Let U_0 and U_1 be the stabilizers of x_0 and x_1 in G . Since G acts transitively on X_0 and on X_1 , we have $G/U_0 \cong X_0$ and $G/U_1 \cong X_1$.

We can view the normed $*$ -algebra $C_c(U_0 \backslash G/U_0)$ as a space of finitely supported functions on X_0 . Since U_0 acts transitively on every sphere around x_0 , it is well-known that the pair (G, U_0) is a Gelfand pair, that is, the algebra $C_c(U_0 \backslash G/U_0)$ is commutative (see for instance [BLRW09, Lemma 2.1]). Observe that $C_c(U_0 \backslash G/U_0)$ is the linear span of the characteristic functions $\delta_n^{(0)}$ (lifted to G) of spheres of even radius n around x_0 . Moreover, $C_c(U_0 \backslash G/U_0)$ is generated by $\delta_2^{(0)}$; indeed, this follows from the formulas (see

[BLRW09, Theorem 3.3])

$$\begin{aligned}\delta_4^{(0)} &= \delta_2^{(0)} * \delta_2^{(0)} - k_0(k_1 - 1)\delta_0^{(0)} - (k_1 - 2)\delta_2^{(0)} \\ \delta_{2n+2}^{(0)} &= \delta_2^{(0)} * \delta_{2n}^{(0)} - (k_0 - 1)(k_1 - 1)\delta_{2n-2}^{(0)} - (k_1 - 2)\delta_{2n}^{(0)} \quad \text{for } n \geq 2.\end{aligned}$$

Let $f_0 = \frac{1}{\|\delta_2^{(0)}\|_1} \delta_2^{(0)}$. We claim that f_0 has all the properties listed in Proposition 4.

Indeed, f_0 is a non-negative and U_0 -bi-invariant function on G with $\int_G f_0(x)dx = 1$. Moreover, let H be the closure of the subgroup generated by the support of f_0 . Assume, by contradiction, that $H \neq G$. Then there exists a function in $C_c(U_0 \backslash G / U_0)$ whose support is disjoint from H . This is a contradiction, as the algebra $C_c(U_0 \backslash G / U_0)$ is generated by f_0 . This shows that $H = G$.

Let π be the unitary representation of G on $L_0^2(\Gamma \backslash G)$ defined by right translations. Observe that the space of $\pi(U_0)$ -invariant vectors is $L_0^2(\Gamma \backslash X_0)$. So, we have a $*$ -representation π_{U_0} of $C_c(U_0 \backslash G / U_0)$ on $L^2(\Gamma \backslash X_0, \mu)$, where μ is the measure on the diagram $D = \Gamma \backslash X$, as defined above.

Similar facts are also true for the algebra $C_c(U_1 \backslash G / U_1)$: this is a commutative normed $*$ -algebra, it is generated by the characteristic function $\delta_2^{(1)}$ of the sphere of radius 2 around x_1 , and the representation π of G on $L_0^2(\Gamma \backslash G)$ induces a $*$ -representation π_{U_1} of $C_c(U_1 \backslash G / U_1)$ on $L_0^2(\Gamma \backslash X_1, \mu)$. Likewise, the function $f_1 = \frac{1}{\|\delta_2^{(1)}\|_1} \delta_2^{(1)}$ has all the properties listed in Proposition 4.

Let A_X be the adjacency operator defined on $\ell^2(X)$ by

$$A_X f(x) = \frac{1}{\deg(x)} \sum_{e \in \partial_0^{-1}(x)} f(\partial_1(e)), \quad f \in \ell^2(X).$$

Since A_X commutes with automorphisms of X , it induces an operator A_D on $L^2(VD, \mu)$ given by

$$A_D f(x) = \frac{1}{\text{indeg}(x)} \sum_{e \in \partial_0^{-1}(x)} \frac{\mu(e)}{\mu(x)} f(\partial_1(e)), \quad f \in L^2(VD, \mu),$$

where D is the diagram obtained from the quotient graph $\Gamma \backslash X$. So, $\Delta = I - A_D$, where Δ is the Laplace operator on D .

Let B_D denote the restriction of A_D to the space $L_0^2(VD, \mu)$. It follows that $\lambda(\Delta) > 0$ if and only if 1 does not belong to the spectrum of B_D .

Proposition 3 will be proved, once we have shown the following

Claim: 1 belongs to the spectrum of B_D if and only if 1_G is weakly contained in π .

For this, we consider the squares of the operators A_X and A_D and compute

$$A_X^2 f(x) = \frac{1}{k_0 k_1} \deg(x) f(x) + \frac{1}{k_0 k_1} \sum_{d(x,y)=2} f(y), \quad f \in \ell^2(X).$$

The subspaces $\ell^2(X_0)$ and $\ell^2(X_1)$ of $\ell^2(X)$ are invariant under A_X^2 and the restrictions of A_X^2 to $\ell^2(X_0)$ and $\ell^2(X_1)$ are given by right convolution with the functions

$$\begin{aligned} g_0 &= \frac{1}{k_0 k_1} \delta_e + \left(1 - \frac{1}{k_0 k_1}\right) f_0 \\ g_1 &= \frac{1}{k_0 k_1} \delta_e + \left(1 - \frac{1}{k_0 k_1}\right) f_1, \end{aligned}$$

where δ_e is the Dirac function at the group unit e of G .

It follows that the restrictions of B_D^2 to the subspaces $L_0^2(\Gamma \backslash X_0, \mu)$ and $L_0^2(\Gamma \backslash X_1, \mu)$ coincide with the operators $\pi_{U_0}(g_0)$ and $\pi_{U_1}(g_1)$, respectively.

For $i = 0, 1$, the spectrum $\sigma(\pi_{U_i}(g_i))$ of $\pi_{U_i}(g_i)$ is the set

$$\sigma(\pi_{U_i}(g_i)) = \left\{ \frac{1}{k_0 k_1} + \left(1 - \frac{1}{k_0 k_1}\right) \lambda : \lambda \in \sigma(\pi_{U_i}(f_i)) \right\}.$$

Thus, 1 belongs to the spectrum of $\pi_{U_0}(f_i)$ if and only if 1 belongs to the spectrum of $\pi_{U_0}(g_i)$.

To prove the claim above, assume that 1 belongs to the spectrum of B_D . Then 1 belongs to the spectrum of B_D^2 . Hence 1 belongs to the spectrum of either $\pi_{U_0}(g_0)$ or $\pi_{U_1}(g_1)$ and therefore 1 belongs to the spectrum of either $\pi_{U_0}(f_0)$ or $\pi_{U_1}(f_1)$. It follows from Proposition 4 that 1_G is weakly contained in π .

Conversely, suppose that 1_G is weakly contained in π . Then, again by Proposition 4, 1 belongs to the spectra of $\pi_{U_0}(f_0)$ and $\pi_{U_1}(f_1)$. Hence, 1 belongs to the spectra of $\pi_{U_0}(g_0)$ and $\pi_{U_1}(g_1)$. We claim that 1 belongs to the spectrum of B_D .

Indeed, assume by contradiction that 1 does not belong to the spectrum of B_D , that is, $B_D - I$ has a bounded inverse on $L_0^2(VD, \mu)$. Since 1 belongs to the spectrum of the self-adjoint operator $\pi_{U_0}(g_0)$, there exists a sequence of unit vectors $\xi_n^{(0)}$ in $L_0^2(\Gamma \setminus X_0, \mu)$ with

$$\lim_n \|\pi_{U_0}(g_0)\xi_n^{(0)} - \xi_n^{(0)}\| = 0.$$

As the restriction of B_D^2 to $L_0^2(\Gamma \setminus X_0, \mu)$ coincides with $\pi_{U_0}(g_0)$, we have

$$\begin{aligned} \|\pi_{U_0}(g_0)\xi_n^{(0)} - \xi_n^{(0)}\| &= \|(B_D^2 - I)\xi_n^{(0)}\| \\ &= \|(B_D - I)(B_D + I)\xi_n^{(0)}\| \\ &\geq \frac{1}{\|(B_D - I)^{-1}\|} \|(B_D + I)\xi_n^{(0)}\| \end{aligned}$$

So, $\lim_n \|B_D\xi_n^{(0)} + \xi_n^{(0)}\| = 0$. On the other hand, observe that B_D maps $L_0^2(\Gamma \setminus X_0, \mu)$ to the subspace $L^2(\Gamma \setminus X_1, \mu)$ and that these subspaces are orthogonal to each other. Hence,

$$\|B_D\xi_n^{(0)} + \xi_n^{(0)}\|^2 = \|B_D\xi_n^{(0)}\|^2 + \|\xi_n^{(0)}\|^2$$

This is a contradiction since $\|\xi_n^{(0)}\| = 1$ for all n . The proof of Proposition 3 is now complete. ■

Next, we rephrase Proposition 3 in terms of expander diagrams. Let (D, i, w) be a diagram with finite volume. For a subset S of VD , set

$$E(S, S^c) = \{e \in ED : \partial_0(e) \in S, \partial_1(e) \notin S\}.$$

We say that D is an *expander diagram* if there exists $\varepsilon > 0$ such that

$$\frac{\mu(E(S, S^c))}{\mu(S)} \geq \varepsilon$$

for all $S \subset VD$ with $\mu(S) \leq \mu(D)/2$. The motivation for this definition comes from expander graphs (see [Lubo94]).

We quote from [Mokh03] and [Morg94] the following result which is standard in the case of finite graphs.

Proposition 5 ([Mokh03], [Morg94]) *Let (D, i, w) be a diagram with finite volume. Assume that $\sup_{e \in ED} i(\bar{e})/i(e) < \infty$ and that $\sup_{x \in VD} \text{indeg}(x) < \infty$. The following conditions are equivalent:*

- (i) D is an expander diagram;
- (ii) $\lambda(D) > 0$.

As an immediate consequence of Propositions 3 and 5, we obtain the following result which relates the existence of a spectral gap to an expanding property of the corresponding diagram.

Proposition 6 *Let X be either the k -regular tree X_k or the bi-partite bi-regular tree X_{k_0, k_1} for $k \geq 3$ or $k_0 \geq 3$ and $k_1 \geq 3$. Let G be a closed subgroup of $\text{Aut}(X)$ satisfying both conditions from Proposition 3. Let Γ be a lattice in G and let $D = \Gamma \backslash X$ be the corresponding diagram of finite volume. The following properties are equivalent.*

- (i) *The unitary representation $\rho_{\Gamma \backslash G}$ on $L^2(\Gamma \backslash G)$ has a spectral gap;*
- (ii) *D is an expander diagram.*

3 Proof of Theorem 1

Let $G = \mathbf{G}(\mathbf{k})$ be the \mathbf{k} -rational points of a simple algebraic group \mathbf{G} over a local field \mathbf{k} and let Γ be a lattice in G . As explained in the Introduction, we may assume that \mathbf{k} is non-archimedean and that $\mathbf{k} - \text{rank}(\mathbf{G}) = 1$. By the Bruhat-Tits theory, G acts on a regular or bi-partite bi-regular tree X with one or two orbits. Moreover, the action of G on X is locally ∞ -transitive (see [Chou94, p.33]).

Passing to the subgroup G^+ of index at most two consisting of orientation preserving automorphisms, we can assume that G acts without inversion. Indeed, assume that $L^2(\Gamma \cap G^+ \backslash G^+)$ has a spectral gap. If Γ is contained in G^+ , then $L^2(\Gamma \backslash G)$ has a spectral gap since G^+ has finite index (see [BeCo08, Proposition 6]). If Γ is not contained in G^+ , then $\Gamma \cap G^+ \backslash G^+$ may be identified as a G^+ -space with $\Gamma \backslash \Gamma G^+ = \Gamma \backslash G$. Hence, 1_{G^+} is not weakly contained in the G^+ -representation defined on $L^2_0(\Gamma \backslash G)$.

Let X be the Bruhat-Tits tree associated to G . It is shown in [Lubo91, Theorem 6.1] (see also [Baum03]) that Γ has fundamental domain D in X of the following form: there exists a finite set $F \subset D$ such that $D \setminus F$ is a union of finitely many disjoint rays r_1, \dots, r_s . (Recall that a ray in X is an infinite

path beginning at some vertex and without backtracking.) Moreover, for every ray $r_j = \{x_0^j, x_1^j, x_2^j, \dots\}$ in $D \setminus F$, the stabilizer $\Gamma_{x_i^j}$ of x_i^j is contained in the stabilizer $\Gamma_{x_{i+1}^j}$ of x_{i+1}^j for all i .

To prove Theorem 1, we apply Proposition 6. So, we have to prove that D is an expander diagramm.

Choose $i \in \{0, 1, \dots\}$ such that, with

$$D_1 = F \cup \bigcup_{j=1}^s \{x_0^j, \dots, x_i^j\},$$

we have $\mu(D_1) > 1/2$.

Let S be a subset of D with $\mu(S) \leq \mu(D)/2$. Then $D_1 \not\subseteq S$. Two cases can occur.

• *First case:* $S \cap D_1 = \emptyset$. Thus, S is contained in

$$\bigcup_{j=1}^s \{x_{i+1}^j, x_{i+2}^j, \dots\}.$$

Fix $j \in \{1, \dots, s\}$. Let $i(j) \in \{0, 1, \dots\}$ be minimal with the property that $x_{i(j)+1}^j \in S$. Then $e_j := (x_{i(j)+1}^j, x_{i(j)}^j) \in E(S, S^c)$. Observe that $|\Gamma_{x_{i+1}^j}| = \deg(x_l^j) |\Gamma_{x_l^j}|$ for all $l \geq 0$. Let k be the minimal degree for vertices in X (so, $k = \min\{k_0, k_1\}$ if $X = X_{k_0, k_1}$). Then $\mu(x_{l+1}^j) \leq \mu(x_l^j)/k$ for all l and

$$\mu(e_j) = \frac{1}{|\Gamma_{e_j}|} \geq \frac{k}{|\Gamma_{x_{i(j)}^j}|} = k\mu(x_{i(j)}^j).$$

Therefore, we have

$$\begin{aligned}
\frac{\mu(E(S, S^c))}{\mu(S)} &\geq \frac{\sum_{j=1}^s \mu(e_j)}{\sum_{j=1}^s \mu(\{x_{i(j)+1}^j, x_{i(j)+1}^j, \dots, \})} \\
&\geq k \frac{\sum_{j=1}^s \mu(x_{i(j)}^j)}{\sum_{j=1}^s \sum_{l=0}^{\infty} \mu(x_{i(j)+l}^j)} \\
&\geq k \frac{\sum_{j=1}^s \mu(x_{i(j)}^j)}{\sum_{j=1}^s \mu(x_{i(j)}^j) \sum_{l=0}^{\infty} k^{-l}} \\
&= k \frac{\sum_{j=1}^s \mu(x_{i(j)}^j)}{\frac{1}{1-k^{-1}} \sum_{j=1}^s \mu(x_{i(j)}^j)} \\
&= k \frac{1}{\frac{1}{1-k^{-1}}} = k - 1.
\end{aligned}$$

•*Second case:* $S \cap D_1 \neq \emptyset$. Then there exist $x \in S \cap D_1$ and $y \in D_1 \setminus S$. Since D_1 is a connected subgraph, there exists a path (e_1, e_2, \dots, e_n) in ED_1 from x to y . Let $l \in \{1, \dots, n\}$ be minimal with the property $\partial_0(e_l) \in S$ and $\partial_1(e_l) \notin S$. Then $e_l \in E(S, S^c)$. Hence, with $C = \min\{\mu(e) : e \in ED_1\} > 0$, we have

$$\frac{\mu(E(S, S^c))}{\mu(S)} \geq \frac{C}{\mu(D)}.$$

This completes the proof of Theorem 1. ■

4 Proof of Theorem 2

Let (D, i, μ) be a k -regular diagram. By the “inverse Bass–Serre theory” of groups acting on trees, there exists a lattice Γ in $G = \text{Aut}(X_k)$ for which $D = \Gamma \backslash X_k$. Indeed, we can find a finite grouping of (D, i) , that is, a graph of finite groups $\mathbf{D} = (D, \mathcal{D})$ such that $i(e)$ is the index of \mathcal{D}_e in $\mathcal{D}_{\partial_0 e}$ for all $e \in ED$. Fix an origin x_0 . Let $\Gamma = \pi_1(\mathbf{D}, x_0)$ be the fundamental group of (\mathbf{D}, x_0) . The universal covering of (\mathbf{D}, x_0) is the k -regular tree X_k and the diagram D can be identified with the diagram associated to $\Gamma \backslash X_k$. For all this, see (2.5), (2.6) and (4.13) in [BaLu01].

In view of Proposition 6, Theorem 2 will be proved once we present examples of k -regular diagrams with finite volume which are not expanders.

An example of such a diagram appears in [Mokh03, Example 3.4]. For the convenience of the reader, we review the construction.

Fix $k \geq 3$ and let $q = k - 1$. For every integer $n \geq 1$, let D_n be the finite graph with $2n + 1$ vertices:

$$\begin{array}{ccccccc} \circ & - & \circ & - & \circ & - \cdots - & \circ & - & \circ \\ x_1^{(n)} & & x_2^{(n)} & & & & x_{2n}^{(n)} & & x_{2n+1}^{(n)} \end{array}$$

Let D be the following infinite ray:

$$\begin{array}{ccccccccccc} \circ & - & \circ & - & D_1 & - & \circ & - & \circ & - & D_2 & - & \circ & - & \circ & - \cdots - & \circ & - & \circ & - & D_n & - & \circ & - & \circ & - \cdots \\ x_0 & & x_1 & & & & x_2 & & x_3 & & & & x_{2n-2} & & x_{2n-1} & & & & & & & & & & & \end{array}$$

We first define a weight function i_n on ED_n as follows:

- $i_n(e) = 1$ if $e = (x_1^{(n)}, x_2^{(n)})$ or $e = (x_2^{(n)}, x_1^{(n)})$
- $i_n(e) = q$ if $e = (x_m^{(n)}, x_{m+1}^{(n)})$ for m even
- $i_n(e) = 1$ if $e = (x_m^{(n)}, x_{m+1}^{(n)})$ for m odd
- $i_n(e) = q$ if $e = (x_{m+1}^{(n)}, x_m^{(n)})$ for m even
- $i_n(e) = 1$ if $e = (x_{m+1}^{(n)}, x_m^{(n)})$ for m odd.

Observe that $i_n(e)/i_n(\bar{e}) = 1$ for all $e \in ED_n$. Define now a weight function i on ED as follows:

- $i(e) = q + 1$ if $e = (x_0, x_1)$
- $i(e) = q$ if $e = (x_1, x_0)$
- $i(e) = 1$ if $e = (x_m, x_{m+1})$ for $m \geq 1$
- $i(e) = q$ if $e = (x_{m+1}, x_m)$ for $m \geq 1$
- $i(e) = i_n(e)$ if $e \in ED_n$.

One readily checks that, for every vertex $x \in D$,

$$\sum_{e \in \partial_0^{-1}(x)} i(e) = q + 1 = k,$$

that is, (D, i) is k -regular. The measure $\mu : VD \rightarrow \mathbf{R}^+$ corresponding to i (see the remark at the beginning of Section 2) is given by

- $\mu(x_0) = 1/(q+1)$
- $\mu(x_{2m-2}) = 1/q^{m-1}$ for $m \geq 2$
- $\mu(x_{2m-1}) = 1/q^m$ for $m \geq 1$
- $\mu(x) = 1/q^n$ if $x \in D_n$.

One checks that, if we define $\mu(e) = i(e)\mu(\partial_0 e)$ for all $e \in ED$, we have $\mu(\bar{e}) = \mu(e)$. Moreover,

$$\mu(D_n) = (2n+1)\frac{1}{q^n}$$

and hence

$$\mu(D) \leq \frac{1}{q+1} + 2 \sum_{n \geq 0} \frac{1}{q^n} + \sum_{n \geq 1} \mu(D_n) < \infty.$$

We have also

$$E(D_n, D_n^c) = \{(x_{2n-1}, x_{2n-2}), (x_{2n}, x_{2n+1})\},$$

so that

$$\mu(E(D_n, D_n^c)) = q \frac{1}{q^n} + \frac{1}{q^n} = \frac{q+1}{q^n}.$$

Hence

$$\frac{\mu(E(D_n, D_n^c))}{\mu(D_n)} = \frac{\frac{q+1}{q^n}}{(2n+1)\frac{1}{q^n}} = \frac{q+1}{2n+1}$$

and

$$\lim_n \frac{\mu(E(D_n, D_n^c))}{\mu(D_n)} = 0.$$

Observe that, since $\lim_n \mu(D_n) = 0$, we have $\mu(D_n) \leq \mu(D)/2$ for sufficiently large n . This completes the proof of Theorem 2. ■.

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